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# Modes in Coupled Optical Resonators with Active Media

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*Summary*—**A general method is proposed to analyze the properties of optical systems composed of several coupled resonators. It is shown that by using appropriate matrices to represent the fields in the resonators and the couplings between them, an equation can be written, often by inspection, for the eigenvalue $s=\sigma+j\omega$ which gives the frequency and the rate of growth of the fields for all the modes of a given system.**

**A re-entrant coupled system with loss and gain regions is discussed as an example. The effects of changes in mirror transmission, resonator length and medium properties are studied using the method.**

## I. INTRODUCTION

IT HAS BEEN suggested by various authors[1,2] that optical masers with desirable properties could be obtained by operating the active medium in a system of coupled optical resonators rather than in simple structures of the Fabry-Perot type. In the latter many modes of oscillation are allowed, separated by equal frequency intervals and with relative growth rates which depend only on the active medium and not on the resonator. Simultaneous oscillation in several modes is thus possible within the linewidth of usual materials. Coupled systems can have modes with unevenly spaced frequencies and different rates of growth or of attenuation if lossy materials are used as well as active media. These properties depend on the resonant system as well as the materials, and are determined by mechanical design and adjustment. Possible features include selective mode suppression, so that only one mode oscillates, or so that only two oscillate with controllable frequency separation.

The properties of arbitrary coupled resonator systems can be studied as an eigenvalue problem. The time-dependence properties of each mode are given by the complex exponential exp $(st)$, where $s=\sigma+j\omega$ expresses the frequency of oscillation and the rate of growth or decay of the mode. The form of the eigenvalue equation specifies the freedom one has to choose the eigenvalues $s$; that is, it shows the possibilities and limitations of each particular system and also of coupled systems in general. The analysis is greatly simplified by considering that the traveling waves inside the resonators are approximately TEM. Use can thus be made of equivalent circuits with coupled TEM transmission lines.

## II. TWO-MIRROR RESONATORS

To introduce the method proposed, we consider first a simple, uncoupled system. Fig. 1(a) shows a resonator formed by two identical mirrors in a medium assumed

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[1] D. A. Kleinman and P. Kisliuk, "Discrimination against unwanted orders in the Fabry-Perot resonator," *Bell Sys. Tech. J.*, vol. 41, p. 453; 1962.

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(a)

(b)

Fig. 1—Two-mirror resonator and its transmission line equivalent representation.

without absorption. Fox and Li[3] have shown there exist modes characterized by field distributions which reproduce themselves in pattern after bouncing between the mirrors. If $x$ and $y$ are (curvilinear) coordinates on the mirrors, and $\psi(xy)$ is the field ($E$ or $H$) leaving one mirror, then the condition for $\psi$ to belong to one of these modes is that the incoming field distribution on the other mirror equals

$$\frac{1}{\gamma}\psi(xy). \tag{1}$$

The constant $\gamma$ and the function $\psi(xy)$ characterize each mode. Both depend on the frequency $\omega$ but, over the narrow frequency range typical of the operation of masers, the $\psi$'s are so close to transverse electromagnetic that they can be assumed independent of $\omega$. The same is true of the magnitudes $|\gamma|$. The phases of $\gamma$ vary linearly with $\omega$. Therefore

$$\gamma(\omega) = |\gamma| e^{j(\theta+\omega T)} \tag{2}$$

where $T$ is the one-way transit time between mirrors and $\theta$ is a constant phase shift. Since $\psi(xy)$ depends on the mode only, the fields leaving one mirror can be represented by a complex number $A$ and the fields arriving to the other by another number $A'$, both quantities being defined so that $|A|^2$ and $|A'|^2$ equal the corresponding powers.

If the medium between the mirrors has linear absorption properties (positive or negative), a time absorption coefficient $\alpha$ can be defined so the fields produced by $\psi(xy)$ are given by

$$e^{-\alpha T}\frac{1}{\gamma}\psi(xy) \tag{3}$$

instead of (1).

Each mode in a two-mirror system is then characterized by a complex propagation coefficient $p$ defined by

$$p = \frac{A'}{A} = \frac{e^{-\alpha T}}{\gamma} = \frac{1}{|\gamma|}e^{-\alpha T - j(\theta+\omega T)}. \tag{4}$$

With an active medium of sufficiently negative $\alpha$, we can have self-replicating patterns which grow in time. Under this oscillation situation, the time dependence of all fields is given by a certain exp ($st$), with $s=\sigma+j\omega$. The condition for resonance is found by following the wave leaving one mirror through two transits and two reflections. Letting $\rho$ be the amplitude reflection coefficient of the mirrors, the condition after the time $2T$ of a round trip is that the phase is the same and the amplitude has increased by a factor exp ($2\sigma T$). Therefore

$$\exp(2\sigma T) = (p\rho)^2 \tag{5}$$

or, substituting $p$ from (4) and using a determinantal form,

$$\begin{vmatrix} -|\gamma| e^{\alpha T + j\theta}e^{sT} & -\rho \\ -\rho & -|\gamma| e^{\alpha T+j\theta}e^{sT} \end{vmatrix} = 0. \tag{6}$$

All quantities are known from the physical and geometrical properties of the system except $s$. The solution for $s$ is

$$\sigma = -\alpha + \frac{\ln\left(\frac{\rho}{|\gamma|}\right)}{T} \tag{7}$$

$$\omega = \frac{-\theta + 2\pi k}{T} \tag{8}$$

$$k = 0, 1, 2, 3, \text{ etc.}$$

The condition for positive $\sigma$ is contained in (7); if $\alpha$ is sufficiently negative for given $|\gamma|$ and $\rho$, oscillation at any of the frequencies given by (8) is possible.

For all the mode solutions characterized by the same $\psi(xy)$, the simple transmission line equivalent circuit of Fig. 1(b) applies. The line has the same length and attenuation as the region between the mirrors, and the terminations have reflectivity $\rho$. This equivalent circuit and (6) are easy to generalize for the case of mirrors with different $\rho$.

## III. Multiresonator Systems

The approach of the preceding section can be extended to coupled configurations. Fig. 2(a) shows a system with three parallel plane mirrors, the middle one partly transparent. In this case, the fields in both parts will be interdependent and the whole structure will behave as a system with modes described by values of $s$ different than for either part alone. To get a coupling of this type, the modes in both regions must have the same $\psi(xy)$. In Fig. 2(b) a single pair of mirrors support several modes of the type described in Section II, with different $\psi(xy)$, but they are coupled by a perturbing irregularity shown diagrammatically on one mirror. Fig. 2(c) is similar to 2(a) but with another coupling arrangement.

[3] A. G. Fox and T. Li, "Resonant modes in a maser interferometer," *Bell Sys. Tech. J.*, vol. 40, p. 453; 1961.

All these systems are formed by reflecting surfaces immersed in media with linear absorption or emission properties, possibly different for each region. The fields these structures can support are all described in terms of self-reproducing patterns $\psi_i(xy)$ traveling from one reflecting surface to another. Calling these propagating patterns traveling waves, we use the symbols $A_i$ and $A_i'$ as before to indicate amplitude and phase for each one of them at the sending and receiving surfaces, respectively. The term mode is used for the whole system.

The arrows on Fig. 2 indicate the traveling waves for each case. Only two pairs are shown for (b) although there could be many. In Fig. 3, a transmission line equivalent circuit is shown for each system of Fig. 2, with the traveling waves numbered to correspond.

To apply the method of Section II, let $[A]$ and $[A']$ be column matrices of order $N$ equal to the number of traveling waves in the system

$$[A] = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_N \end{bmatrix} \tag{9}$$

$$[A'] = \begin{bmatrix} A_1' \\ A_2' \\ \cdot \\ \cdot \\ \cdot \\ A_N' \end{bmatrix}. \tag{10}$$

The reflecting-transmitting surfaces perform the function of coupling the matrices $[A]$ and $[A']$; each element of $[A]$ is a linear combination of the elements of $[A']$. So, we can define a square coupling matrix $[C]$, of order $N$, such that

$$[A] = [C][A']. \tag{11}$$

The matrix $[C]$ is not necessarily symmetric, and its elements will be complex if fixed phase delay is introduced at the coupling. As no traveling wave couples to itself, the diagonal elements are all zero: $C_{ii}=0$. For passive couplings,

$$\sum_j |C_{ij}|^2 \leq 1. \tag{12}$$

To study the time dependence of the fields under resonant conditions we write another relationship between $[A]$ and $[A']$,

$$[A'] = [P][A]. \tag{13}$$

The square propagation matrix $[P]$ has only diagonal elements $P_{ii}$ as no interaction is assumed between traveling waves except at the surfaces:

$$A_i' = P_{ii}A_i. \tag{14}$$

Each $P_{ii}$ is not equal to the $p_i$ defined in (4) because $A_i$ and $A_i'$ in (14) are considered at the same instant of time $t$, while according to the definition

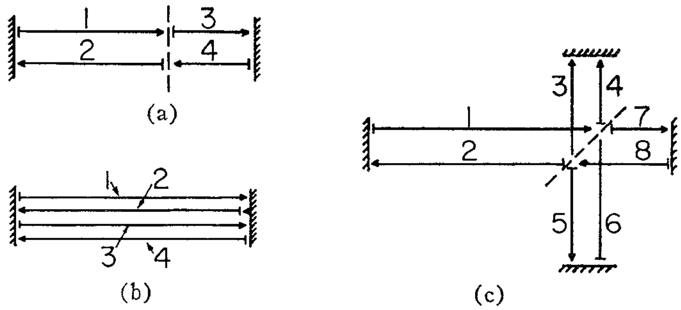


Fig. 2—Examples of coupled resonator systems. The broken lines show transmitting-reflecting surfaces. The arrows indicate the traveling waves; only two pairs are shown in (b).

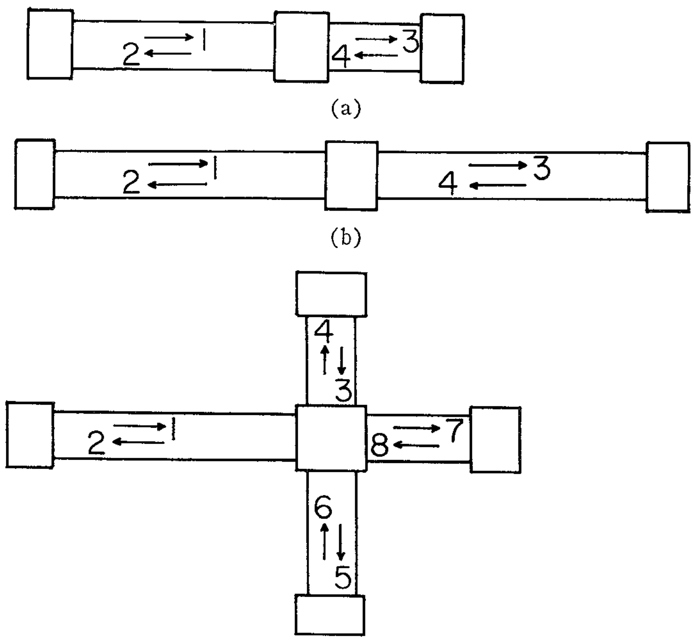


Fig. 3—Equivalent transmission line circuits for the systems of Fig. 2. The traveling waves are numbered to correspond. The boxes represent passive couplers.

$$p_i = \frac{A_i'(t)}{A_i(t - T_i)}. \tag{15}$$

Since all fields grow with exp $(\sigma t)$,

$$P_{ii} = \frac{A_i'(t)}{A_i(t)} = p_i e^{-\sigma T_i} \tag{16}$$

where, as before,

$$p_i = \frac{1}{|\gamma_i|} e^{-\alpha_i T_i - j(\omega T_i + \theta_i)}. \tag{17}$$

Combining (11) and (13) we have

$$[A] = [C][P][A] \tag{18}$$

or

$$\{[C][P] - [I]\}A = 0 \tag{19}$$

where $[I]$ is the identity matrix.

Each solution $[A]$ of this equation gives the starting amplitudes and phases of all traveling waves corresponding to a mode of the system. The eigen value is $s=\sigma+j\omega$, which is the only quantity in the curly bracket not determined directly by the geometry and physical properties of the system. The values of $s$ are given by the following determinantal equation, obtained by performing the operations and using the fact that the $p_i$'s are nonzero:

$$D = \begin{vmatrix} -K_1 e^{sT_1} & C_{12} & \cdots & C_{1N} \\ C_{21} & -K_2 e^{sT_2} & \cdots & C_{2N} \\ C_{N1} & C_{N2} & \cdots & -K_N e^{sT_N} \end{vmatrix} = 0 \quad (20)$$

where

$$K_i = |\gamma_i| e^{\alpha_i T_i + j\theta_i}. \quad (21)$$

$|K_i|^2$ is the fractional power lost per one-way pass through the $i$th resonator; if $|K_i|<1$, *i.e.*, if $\alpha_i$ is sufficiently negative to compensate for $|p_i|>1$, there is net gain in the $i$th traveling wave.

## IV. Solution of the Determinantal Equation

For any given system, (20) is readily written. The diagonal elements are given by the parameters $\alpha$, $|\gamma|$, $\theta$ and $T$ of each resonator, and the coupling coefficients $C_{ij}$ by the system geometry and mirror properties.

To find $s$, we must solve a transcendental equation. In many cases, however, the determinant reduces to a polynomial with integer exponents. This happens when all the transit times $T_i$ are exact multiples of some time $T_0$,

$$T_i = m_i T_0. \quad (22)$$

Then we can use a new complex variable instead of $s$,

$$z = e^{sT_0}, \quad (23)$$

and write

$$e^{sT_i} = z^{m_i} \quad (24)$$

so that the determinant in (20) becomes a polynomial $D(z)$ of order

$$M = \sum_i m_i. \quad (25)$$

Consequently, a coupled system will possess $M$ possible values of $z$, distinct or multiple, representable on the complex plane. Their magnitude gives the growth rate $\sigma$ of the corresponding mode: all the $z$'s within the circle $|z|=1$ are for damped modes, and those outside for growing modes. The angle of each $z$ is the frequency $\omega$ of the mode multiplied by the constant $T_0$ and, as the angles are multivalued, there are infinitely many $\omega$'s for each $z$, in agreement with (23), separated by equal intervals $2\pi/T_0$. A complete set of $M$ values of $z$ with the angles taken between 0 and $2\pi$ forms a period for all the possible values of $\omega$ allowed by the system. How many of these periods have to be considered in a practical case depends of course on the frequency dependence of the active media.

The form of the equation $D(z)=0$ gives some properties of the solutions. Multiple roots can exist, but there is always more than one root, as $D(z)$ is not of the form $(z-z_0)^M$.

Eq. (20) also permits the study of small variations in the physical dimensions of the system, such as those caused by moving a mirror by one wavelength or less. The effect of these "tuning" adjustments can be expected to be considerable, as the impedances in the transmission line equivalent may vary drastically with changes in length. If in a system in which (22) applies we introduce a variation of one $T_i$ of the order of a period of oscillation,

$$\Delta T_i \leq 2\pi/\omega,$$

then $D(z)$ is modified by changing the corresponding integer $m_i$ to a value

$$m_i(1+\epsilon)$$

where

$$|\epsilon| = \left| \frac{\Delta T_i}{T_i} \right| \ll 1.$$

So, any term containg $z^{m_i}$ will be multiplied by $z^{\epsilon m_i}$. The magnitude of this quantity is very close to unity and its angle is

$$\epsilon m_i(\omega T_0 + 2\pi K); \qquad K = 0, 1, 2, \text{ etc.}$$

If we vary continuously the value of one $T_i$, the effect on $D(z)$ is a change in angle but not in magnitude of all the coefficients of the terms containing the corresponding $m_i$. A variation $\Delta T_i = 2\pi/\omega$ causes a change of $2\pi$ in the angle of the coefficients.

## V. Example

The system shown in Fig. 4 displays characteristics typical of the configurations discussed in this paper, and will be used to illustrate the method of analysis.

The active medium is in a resonator shaped so that both its ends can interact, to an extent determined by the reflectivity of the end walls, through a passive resonator from which output beams are coupled out by means of a tilted plate. A small rotation of this plate will change the path length in air.

The pertinent parameters of the system are the reflectivity and transmissivity of the end mirrors and of the plate, the plate angle, the physical dimensions and the properties of the active medium. The power extracted by the tilted plate can be considered as a loss of the passive resonator and the phase shift as an adjustment of its length, so a proper value for $K$ in the passive resonator accounts for all plate effects. Fig. 5 is a representation showing the four coupled traveling waves.

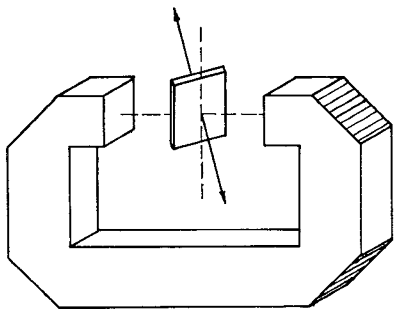

Fig. 4—Example of coupled resonator system. The active medium is in the broken ring-shaped resonator with four mirrors and two partly transparent end surfaces. The variable tilted plate in the air-filled resonator reflects part of the energy away from the system, as shown by the arrows.

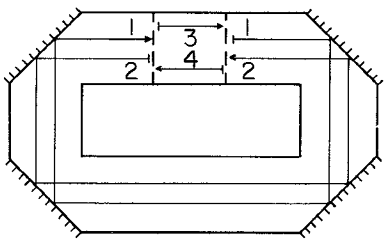


Fig. 5—Diagrammatic representation of the system of Fig. 4.

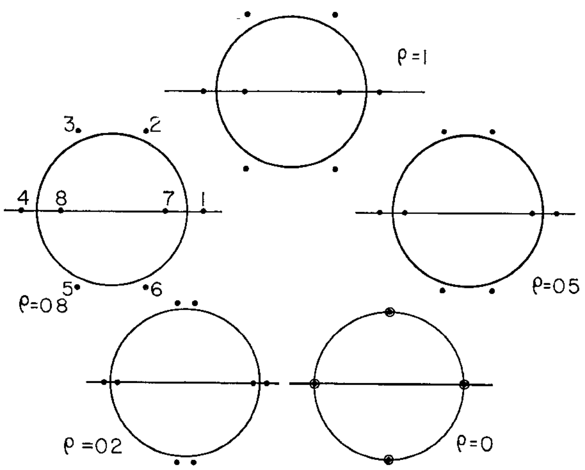


Fig. 6—Mode eigenvalues for the system of Fig. 4, represented in the complex $z$ plane, for several values of the end surface reflectivity $\rho$. The circles have unit radius.

Let us first make the transit time in the active region an exact multiple of that of the passive region,

$$T_1 = T_2 = mT_3 = mT_4 = mT_0. \tag{26}$$

The determinantal equation $D(z)=0$ can be written at once,

$$D(z) = \begin{vmatrix} -K_1z^m & \rho & \tau & 0 \\ \rho & -K_2z^m & 0 & \tau \\ \tau & 0 & -K_3z & -\rho \\ 0 & \tau & -\rho & -K_4z \end{vmatrix} = 0. \tag{27}$$

The symbols $\rho$ and $\tau$ stand for reflectivity and transmissivity respectively. In what follows, we assume zero mirror loss, so

$$\rho^2 + \tau^2 = 1. \tag{28}$$

Expanding (27), writing $K_2=K_1$, $K_4=K_3$ and using (28) we have

$$z^{2m+2} - \frac{2(1-\rho^2)}{K_1K_3} z^{m+1} - \frac{\rho^2}{K_3^2} z^{2m} - \frac{\rho^2}{K_1^2} z^2 + \frac{1}{K_1^2K_3^2} = 0. \tag{29}$$

Putting numbers for $\rho$, $K_1$, $K_3$ and $m$, we can solve for the $2m+2$ values of $z$. Plotted on the complex $z$ plane, these values would show the frequency spacing and growth rates of the $2m+2$ modes in a period of the complete set of solutions, repeating at frequency intervals of $2\pi/T_0$. Changes in the parameters would produce loci of the roots.

The discussion of numerical results is not drastically affected by the assumption that the gain and loss per pass in each resonator are equal, *i.e.*, that there is zero net gain around the complete loop. This simplifies the discussion by eliminating one parameter; let

$$\frac{1}{K_1^2} = K_3^2 = \delta > 1. \tag{30}$$

In terms of $\delta$, (29) becomes

$$\sinh \tfrac{1}{2}(m+1)T_0s = \pm \rho \sinh \tfrac{1}{2}[(m-1)T_0s - \ln \delta] \tag{31}$$

and, if $\delta$ is only slightly larger than unity, as in typical practical cases, the last equation separates so only the $\sigma$

TABLE I

| Root Number | $\rho=1$ | $\rho=0.8$ | $\rho=0.5$ | $\rho=0.2$ | $\rho=0$ |
|---|---|---|---|---|---|
| 1 | $0.167+j0°$ | $0.143+j0°$ | $0.100+j0°$ | $0.045+j0°$ | $0+j0°$ |
| 2 | $0.167+j60°$ | $0.097+j66°$ | $0.034+j75°$ | $0.005+j84°$ | $0+j90°$ |
| 3 | $0.167+j120°$ | $0.097+j114°$ | $0.034+j105°$ | $0.005+j96°$ | $0+j90°$ |
| 4 | $0.167+j180°$ | $0.143+j180°$ | $0.100+j180°$ | $0.045+j180°$ | $0+j180°$ |
| 5 | $0.167-j120°$ | $0.097-j114°$ | $0.034-j105°$ | $0.005-j96°$ | $0-j90°$ |
| 6 | $0.167-j60°$ | $0.097-j66°$ | $0.034-j75°$ | $0.005-j84°$ | $0-j90°$ |
| 7 | $-0.500+j0°$ | $-0.333+j0°$ | $-0.167+j0°$ | $-0.055+j0°$ | $0+j0°$ |
| 8 | $-0.500+j180°$ | $-0.333+j180°$ | $-0.167+j180°$ | $-0.055+j180°$ | $0+j180°$ |

but not the $\omega$ of the modes depends on $\delta$ as follows:

$$\sin \tfrac{1}{2}(m+1)T_0\omega = \pm \rho \sin \tfrac{1}{2}(m-1)T_0\omega \quad (32)$$

$$\frac{\ln \delta}{T_0\sigma} = (m-1) - (m+1)\left\{\frac{\tan \frac{1}{2}(m-1)T_0\omega}{\tan \frac{1}{2}(m+1)T_0\omega}\right\}. \quad (33)$$

When $\rho = 0$, there is no reflection and the solutions are $m+1$ roots equally spaced around the unit circle of the $z$ plane, all double because there is a wave in each loop direction for each frequency. When $\rho = 1$, the two resonators act independently and the roots form two sets: a pair inside the unit circle on the positive and negative real axis and a set of $2m$ single roots equally spaced on a circle with radius larger than 1. For intermediate values of $\rho$ the roots have intermediate positions.

Fig. 6 shows the root positions for $m = 3$ and constant $\delta$, corresponding to various $\rho$'s. As $\rho$ decreases from 1, *i.e.*, as the two resonators are increasingly coupled, the roots all get closer to the unit circle, but the ratio of the positive $\sigma$'s increases; this shows that in practice we can expect an increase of threshold but a decrease in number of modes actually in oscillation. The value of the quantity

$$\frac{T_0\sigma}{\ln \delta} + jT_0\omega$$

is given in Table I for each root. When going from $\rho = 0.8$ to $\rho = 0.2$, the largest positive $\sigma$ becomes 3.1 times smaller but the ratio of this $\sigma$ to the next increases by a factor of 6.25.

Let us study now the effect of changing the lossy region length by amounts comparable with the wavelength. According to Section IV, the locus of the roots of $D(z) = 0$ is found by multiplying $K_3$ in (29) by exp $(j\phi)$, where $\phi$ varies from 0 to $2\pi$. Instead of (31) we have

$$\sinh \tfrac{1}{2}[(m+1)sT_0 + j\phi] = \pm \rho \sinh \tfrac{1}{2}[(m-1)sT_0 - \ln \delta - j\phi], \quad (34)$$

which separates as before into

$$\sin \tfrac{1}{2}[(m+1)T_0\omega + \phi] = \pm \rho \sin \tfrac{1}{2}[(m-1)T_0\omega - \phi] \quad (35)$$

and

$$\frac{\ln \delta}{T_0\sigma} = (m-1) - (m+1)\left\{\frac{\tan \frac{1}{2}[(m-1)T_0\omega - \phi]}{\tan \frac{1}{2}[(m+1)T_0\omega + \phi]}\right\}. \quad (36)$$

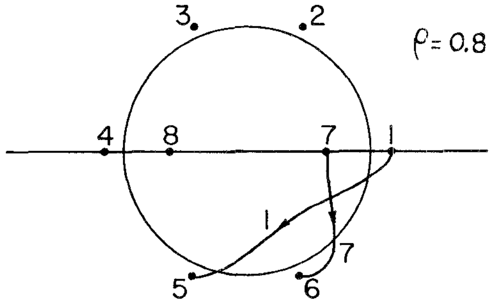


Fig. 7—Locus of the roots numbered 1 and 7 in Fig. 6 when the tilted plate angle is changed. The other roots describe complementary curves.

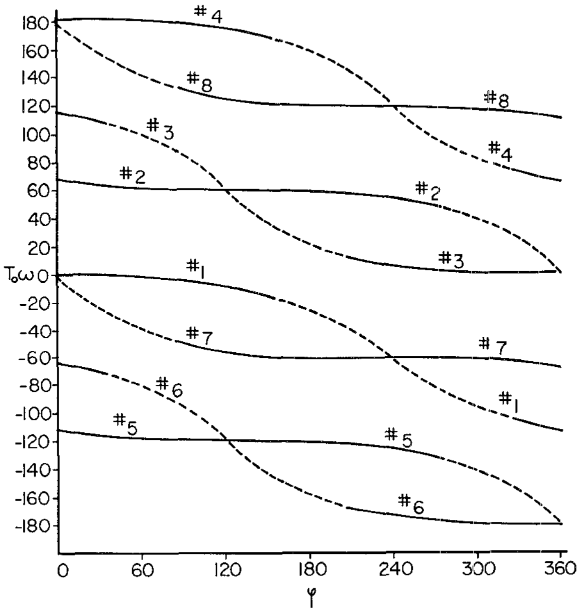


Fig. 8—Normalized frequencies of the modes of the system shown in Fig. 4 when the passive resonator length is varied through one wavelength by adjusting the tilted plate.

When $\phi$ is varied, each root describes a locus curve, the whole pattern repeating every $2\pi$. The case of $m = 3$ and $\rho = 0.8$ is shown in the next two figures. For clarity, the loci of only two roots are shown on Fig. 7; the others describe complementary curves. Fig. 8 gives the normalized $\omega$'s of all eight roots as a function of $\phi$, with the dotted parts corresponding to the regions where $\sigma < 0$. It can be seen that the frequency intervals between modes and also their growth parameters depend on $\phi$; therefore the number of modes that will appear and the frequency differences between them can be controlled by "tuning" the passive resonator, which also determines the periodicity of the solutions.